Numerical Aspects in Reachability Analysis of Nonlinear Hybrid Systems

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The ARIADNE Golden Four requirements

1. Define rigorous mathematical semantics for the analysis of continuous and hybrid systems.
2. Numerical soundness in all operations.
3. Allow arbitrary accuracy by handling nonlinear behavior directly.
4. Allow proving and disproving of properties of a system.

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In the following lecture we will consider these points as requirements, while trying to be as general as possible in respect to implementation.

Outline

Computing on continuous spaces

"Classical" computability theory

- is a function on the natural numbers $f : \mathbb{N}^n \mapsto \mathbb{N}^m$ computable by a Turing Machine?

What happens for *functions on continuous spaces?*

- e.g. function on the reals $f : \mathbb{R}^n \mapsto \mathbb{R}^m$
- how do we represent inputs and outputs?
- how are computations performed?
- which classes of functions are computable? And which are not?

Computable Analysis

A different notion of computability

- Introduced by Klaus Weihrauch and co-workers
- Computation is performed by Turing Machines acting on infinite streams of data
- Data streams encode a sequence of approximations to some quantity
- A function is computable in this theory if:

  given a data stream encoding a sequence of approximations converging to the input

  it is possible to calculate a data stream encoding a sequence of approximations converging to the output
- Finite computations are obtained by terminating when a given accuracy criterion is satisfied:
  - computable functions can be approximated to any desired accuracy

A simple problem

Let $p(x)$ be a polynomial with rational coefficients:
is $p(x) = 0$?

- Classical computability: if x is a rational, then the problem is decidable.
- Computable analysis: if x is a real number, then the problem is semi-decidable:
  - when $p(x) \neq 0$ we can find a sufficiently accurate $\tilde{x}$ to give a negative answer
  - when $p(x) = 0$, no matter how accurate $\tilde{x}$ is, we cannot exclude the possibility that $p(x) \neq 0$, and thus we cannot give a positive answer

The fundamental theorem

Only continuous functions are computable, with respect to a given representation for the data and to the corresponding topology

- a necessary (but not sufficient) condition:
  - if a function is discontinuous, then it is uncomputable
  - a continuous function may be uncomputable
- The choice of the representation is essential:
  - we can make a function computable by requiring more information on the inputs, and/or less information on the outputs

Are hybrid automata computable?

Theorem (Collins 2011)

For any coherent semantics of evolution, the finite-time reachable set of a hybrid automaton is uncomputable.

- Discrete transitions can cause discontinuities in both space and time, even for simple systems
- By the fundamental theorem of computable analysis, this means that the reachable set of hybrid automata is, in general, uncomputable.

Can we recover computability?

- By imposing restrictions on dynamics, reset functions, guards and invariants we can regularize the evolution to make it approximable either from above or from below

. . . however . . .

- the conditions for approximation of the reachable set from above are different from the ones for approximation from below
- we can only obtain a semi-decidable problem

Upper and lower semantics

Definitions

Theorem

Given a Hybrid Automaton with continuous dynamics and reset functions:

Upper semantics *if guards and invariants are closed, then the finite-time reachable set is approximable from above;*

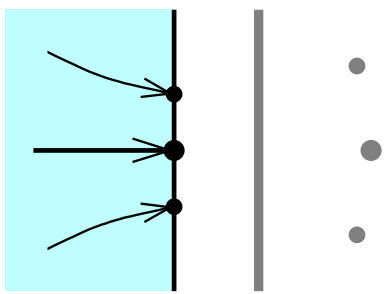
Lower semantics *if guards and invariants are open, then the finite-time reachable set is approximable from below.*

Upper and lower semantics

An example

Consider a location l_0 with invariant $x \leq a$ and a transition that leaves l_0 when $x \geq b$

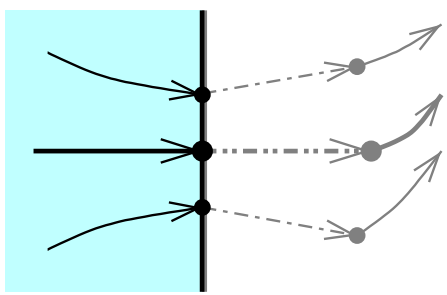
$a < b$



Upper: No Transition

Lower: No Transition

$a = b$



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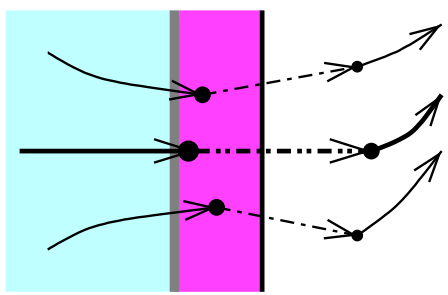

Upper: Transition

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Approximations to the reachable set

Given a hybrid automaton H and an initial set I, it is possible to compute two approximations of the reachable set Re up to a given time t (including the infinite-time case):

- an outer approximation O of the states reached by H starting from I such that:

$$Re \subset O$$

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- an outer approximation O of the states reached by H starting from I such that:

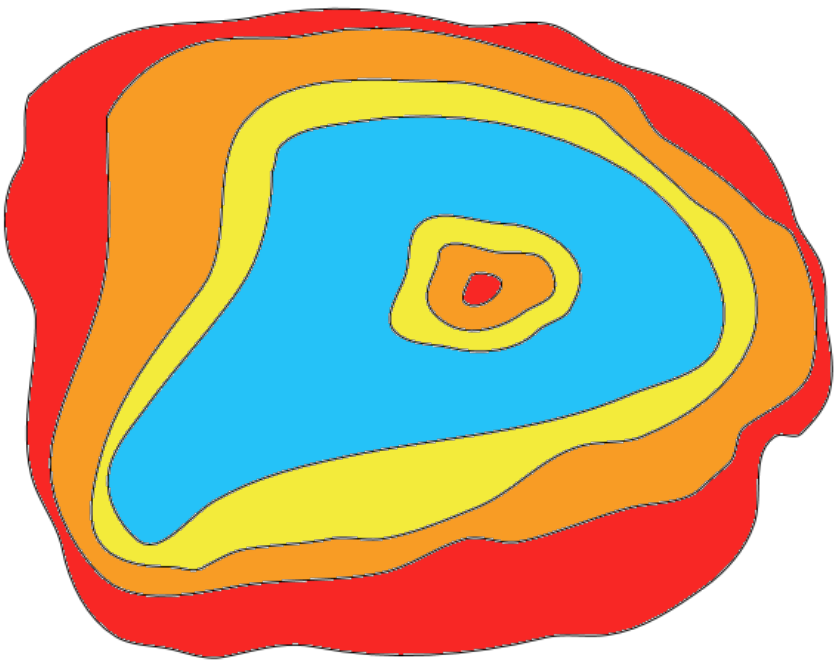
$$Re \subset O$$

- for a given $\varepsilon > 0$, an ε-lower approximation L_ε of the states reached by H starting from I such that:

$$\exists x \in Re \ s.t. \ ||x - L_\varepsilon|| \leq \varepsilon$$

L_ε is an overapproximation of a subset of Re.

Outer approximation O

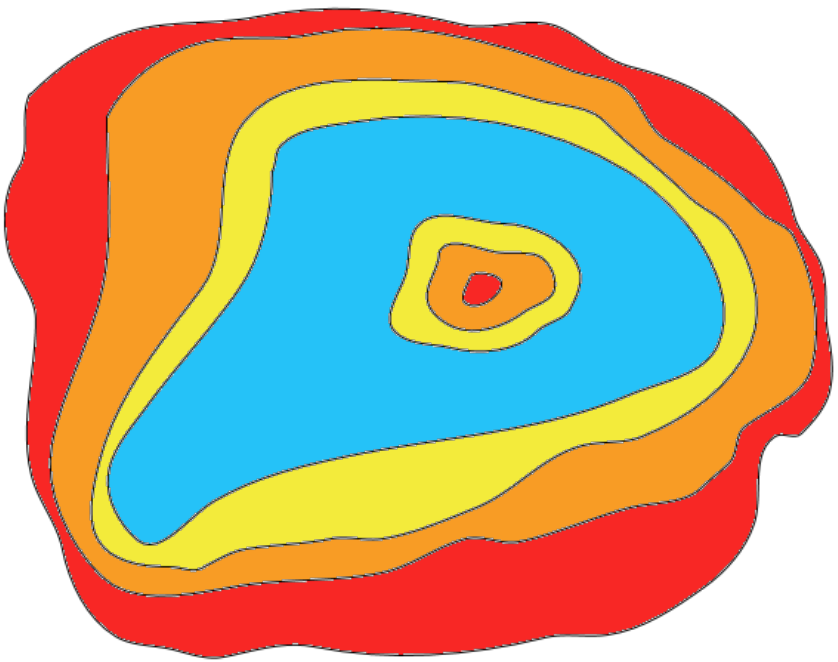


- Blue: reachable set

This is a sequence of approximations from above:

- Red + Orange + Yellow + Blue: coarse O
- Orange + Yellow + Blue: finer O
- Yellow + Blue: finest O

Outer approximation O



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A valid, albeit useless, O is the whole continuous space.

ε-lower approximation L_ε

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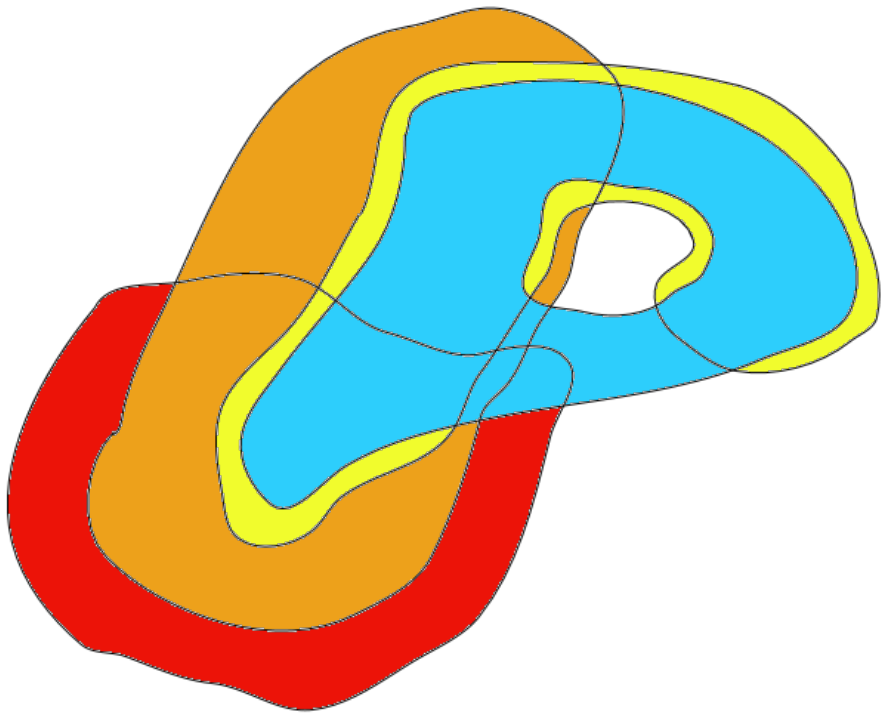
This is a sequence of approximations from below:

- Interior of outline of Red: coarse L_ε
- Interior of outline of Orange: finer L_ε
- Interior of outline of Yellow: finest L_ε



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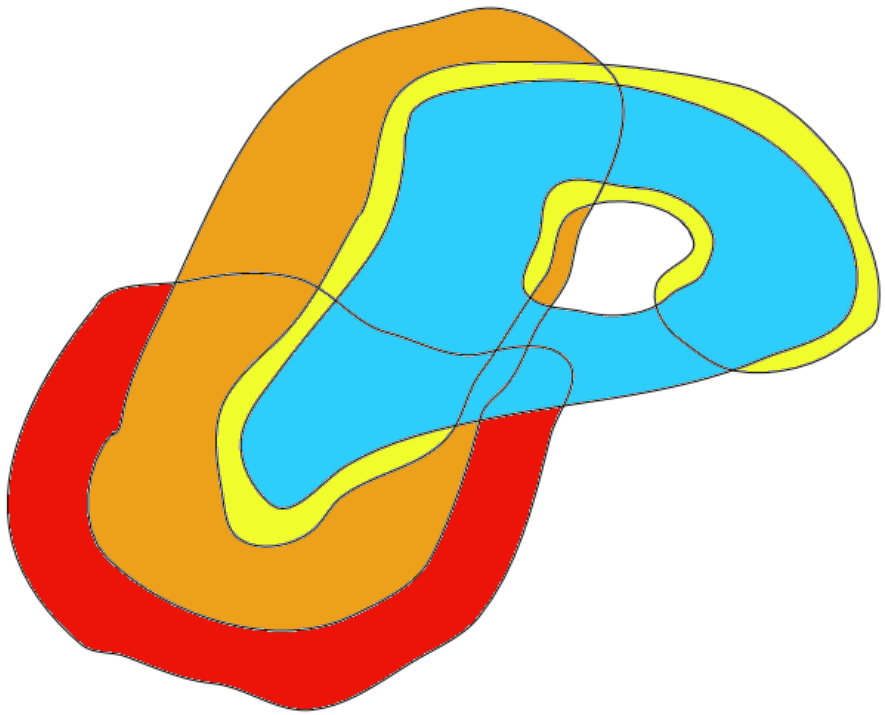
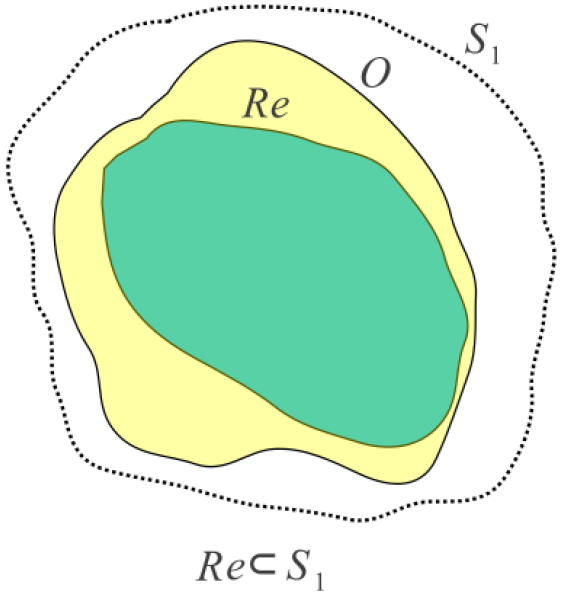
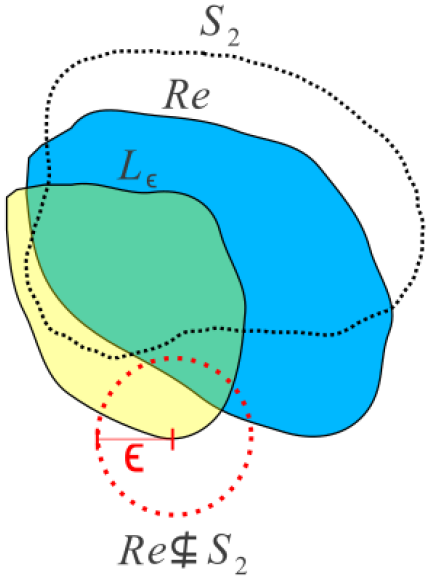


A valid, albeit useless, L_ε is the empty set.

How to use approximations to verify properties

- S_1, S_2 are sets within which a property is satisfied
- $O \subset S_1 \rightarrow Re \subset S_1$
- $||S_2 - L_\varepsilon|| > \varepsilon \rightarrow Re \not\subseteq S_2$





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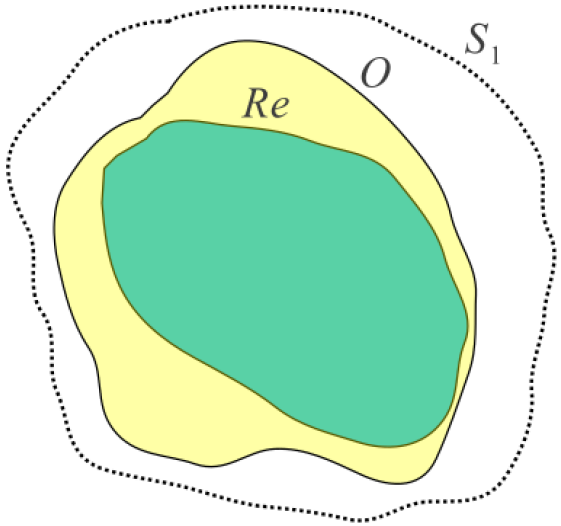

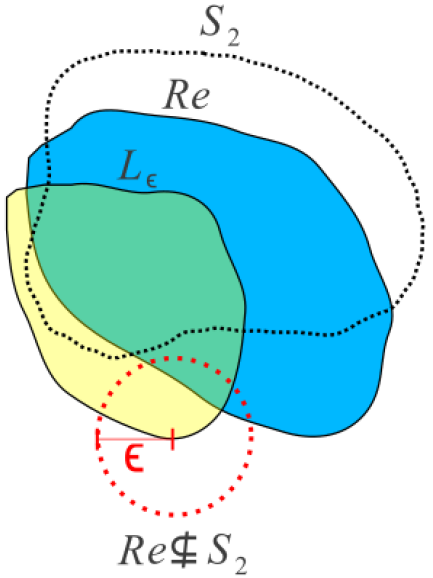

If for a given set of accuracy parameters no answer is found, we can recalculate the approximations with a finer accuracy.

$\rightarrow$ (possibly infinite) sequence of approximations

Switching between representations might be required

- Accurate representations are useful for frequent events (such as continuous steps of evolution), in order to limit accumulation of overapproximation error;

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- Accurate representations are useful for frequent events (such as continuous steps of evolution), in order to limit accumulation of overapproximation error;
- Coarse representations are useful for sporadic events, where operations such as intersection, joining and splitting are required and would be inefficient/ineffective on accurate representations.

The role of functions

Functions can be used to represent Hybrid Automata:

- For every discrete location, a function $Dyn : \mathbb{R}^n \mapsto \mathbb{R}^n$ is used to represent the continuous dynamics.
- Invariants are represented using single-valued functions $Inv : \mathbb{R}^n \mapsto \mathbb{R}$ that are negative exactly when the invariant is true.
- Discrete transitions are represented using a function $Act : \mathbb{R}^n \mapsto \mathbb{R}$ that is positive when the guard of the transition is true (and negative otherwise), and a reset function $Res : \mathbb{R}^n \mapsto \mathbb{R}^n$.

A function along with a finite domain can also specify a region of space for the evolution of an automaton.

Representing functions in the nonlinear case

We represent f from a function mapping a parameter space into the state space: $p : \mathbb{R}^n \mapsto \mathbb{R}^m$, i.e., $\{p_j : \mathbb{R}^n \mapsto \mathbb{R}\}_{j=1}^m$.

- We want $p_j = \sum_{i=0}^{N_j} c_{ij}\beta_{ij}$, i.e., a linear combination of terms in a basis $\{\beta\}$ in the parameter space.

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Finite representation

Since the exact representation of f in general would require infinite terms, and since we need to overapproximate, we add a uniform error term e to the expansion for enclosure.

$\rightarrow$ Ultimately we have $f_j \subset p_j + e_j$ for the j-th component of f.

Advantages and drawbacks of a nonlinear basis

Advantages

- We are not limited to a convex representation;
- We can approximate arbitrarily close by increasing the number of terms and/or the number of parameters (i.e., curve segments in the boundary);
- Algebraic operations between sets use results from Interval Analysis: efficient.

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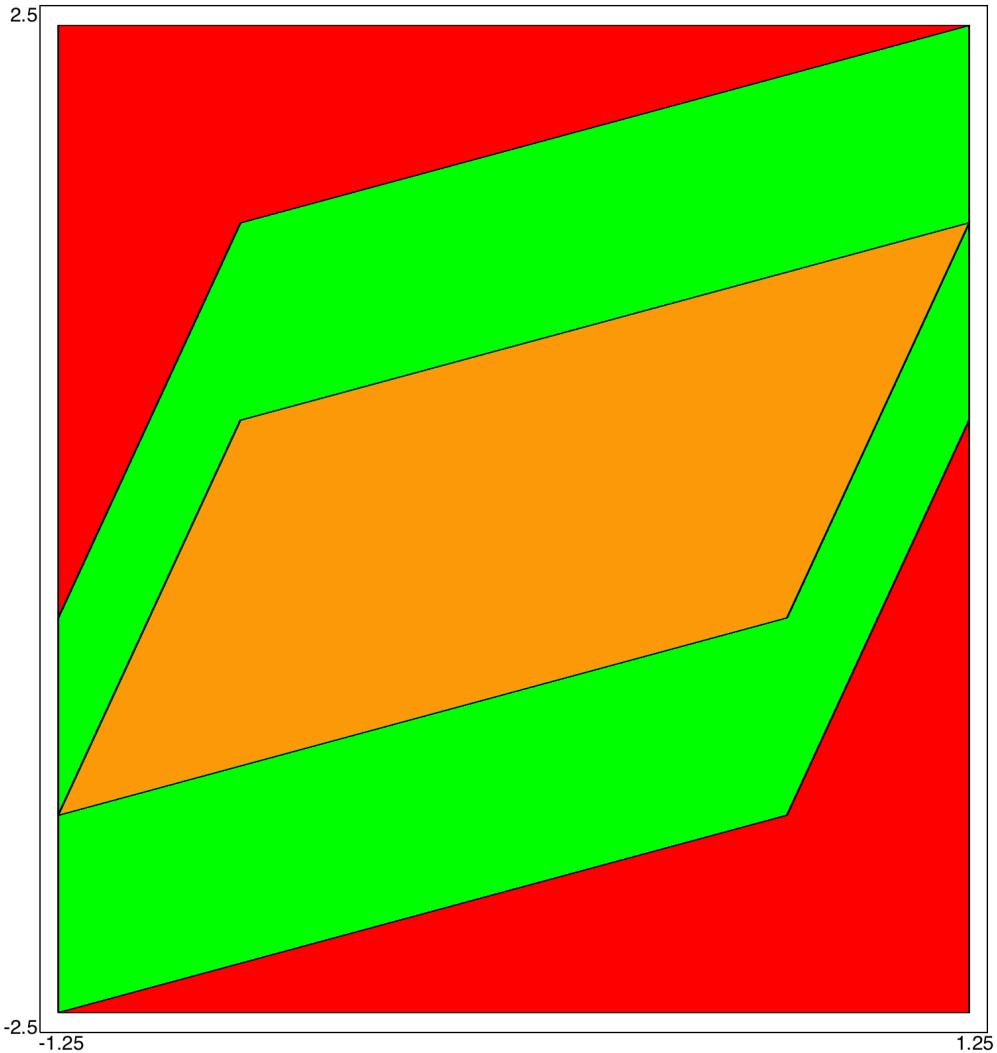
Drawbacks

- Observation of a set is limited: we can efficiently evaluate only the bounds of the function over a domain
  - $\rightarrow$ We need to iteratively split the function to improve evaluation.
- Splitting is done on the domain space: the larger the dimension of the domain space, the more overlapping the resulting split sets.

Sets from Taylor Models: a (linear) example



Set: $[-1,1]^2 \mapsto \mathbb{R}^2$

$x = p_0 + 0.25p_1 \pm 0$

$y = 0.5p_0 + p_1 \pm 0$

Set: $[-1,1]^3 \mapsto \mathbb{R}^2$

$x = p_0 + 0.25p_1 \pm 0$

$y = 0.5p_0 + p_1 + p_2 \pm 0$

or $y = 0.5p_0 + p_1 \pm 1$

Its bounding box: $[-1,1]^2 \mapsto \mathbb{R}^2$

$x = 1.25p_0 \pm 0$

$y = 2.5p_1 \pm 0$

Accuracy control

Numerical parameters available

Allow to decide if a polynomial term should be added into e

1. Maximum polynomial order
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Reconditioning operations

Trade between accuracy and domain space complexity

a. Convert e into an additional parameter $\rightarrow$ increase n

b. Sweep all terms where a parameter appears into $e \rightarrow$ reduce n

Representation of sets using a grid

Definition (Grid)

A coordinate-aligned discrete partitioning of the variables space, which identifies cells of different sizes.

Definition (Grid set)

A marking of cells locked to a grid.

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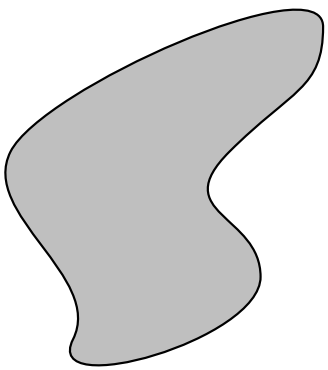

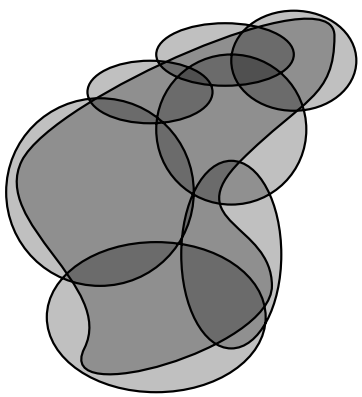

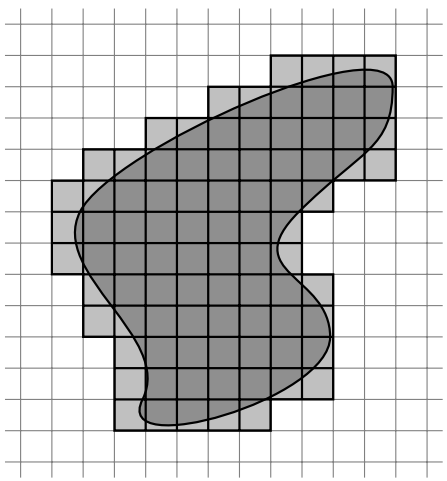

Grid sets - pros and cons

Pros

- Converts easily from/to a polynomial model;
- Allows a compact internal representation, e.g. markings on a binary tree or a binary decision diagram;
- Cells can be split/joined by changing the depth of the markings;
- Union, intersection, difference and inclusion can be performed very efficiently.

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Cons

- They are coarse when using large cell sizes;
- Using small cell sizes is computationally demanding, especially for a large number of variables.

Hybrid evolution of a set

Continuous step

1. From the starting set, given a time step h, construct the flow set
2. Apply to the whole $[0, h]$ time interval to get the reached set
3. Apply to the h time value to get the finishing set

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Discrete step

1. From the flow set, identify the presence of intersections with guard sets
2. Compute crossing times with the guards
3. Compute intersections with the guards
4. Apply the resets

How to construct a flow set

We need to find a set Φ for which the Picard operator applied on itself is a contraction:

$$\Phi_{k+1}(x,t) = x + \int_0^t f(\Phi_k(x,s))ds \subseteq \Phi_k(x,t) \ , \ \forall x \in X_0$$

where f is the ODE function and X_0 is the starting set.

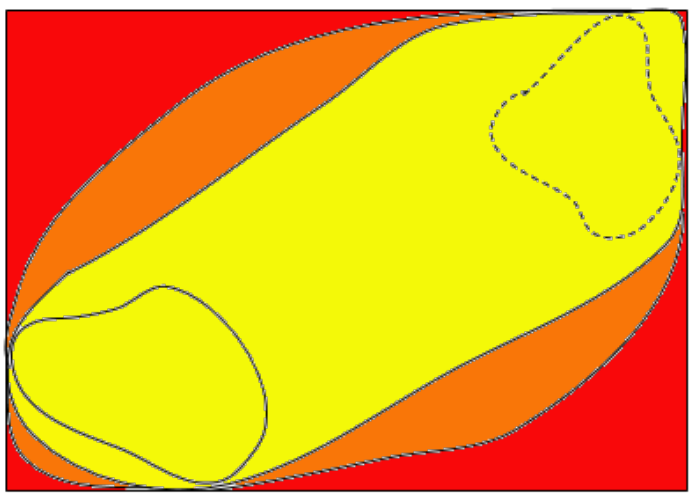
There are two approaches:

1. Get a bounding for Φ and refine Φ until satisfied with the result;
2. Construct a polynomial with desired order for Φ, then find a uniform error e for which the contraction is satisfied.

How to construct a flow set - Method 1

1. Use the Euler Method on the bounding box of X_0, called X_0^{bb}: if the result is contained in X_0^{bb}, then it is a valid Φ_0.
2. Use the Picard operator, integrating using automatic differentiation, to refine Φ from Φ_0 as many times as desired.

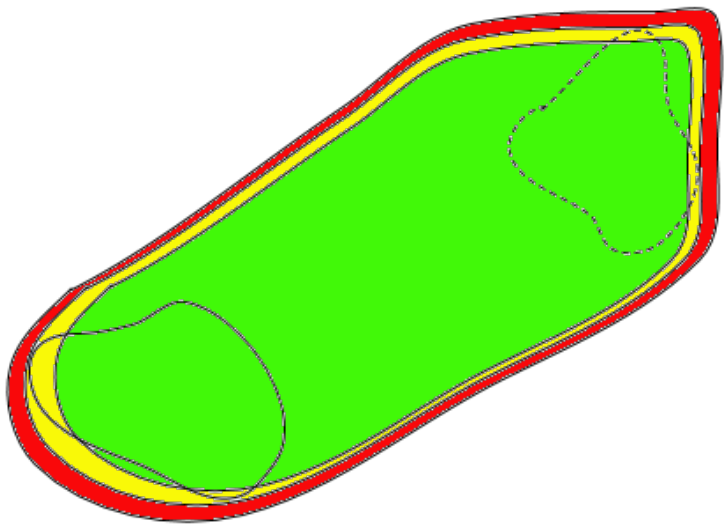
$\rightarrow$ The temporal order increases, the uniform error decreases.



How to construct a flow set - Method 2

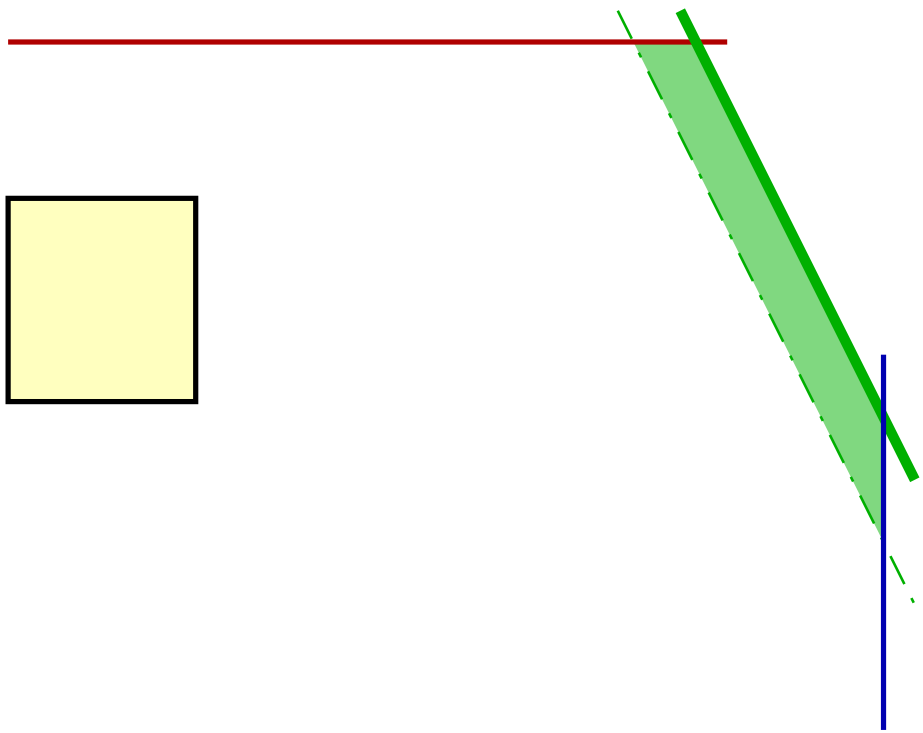
1. Construct a polynomial expansion for Φ called p, using automatic differentiation;
2. Try a value of the uniform error e such that $p + e$ is a contraction:
   - If successful, repeat 2. using the uniform error obtained from the contraction (until satisfied with the result);
   - If not, try with a larger value of e.

$\rightarrow$ The temporal order is fixed, the uniform error decreases.



Finite-time evolution

A sequence of continuous and discrete steps



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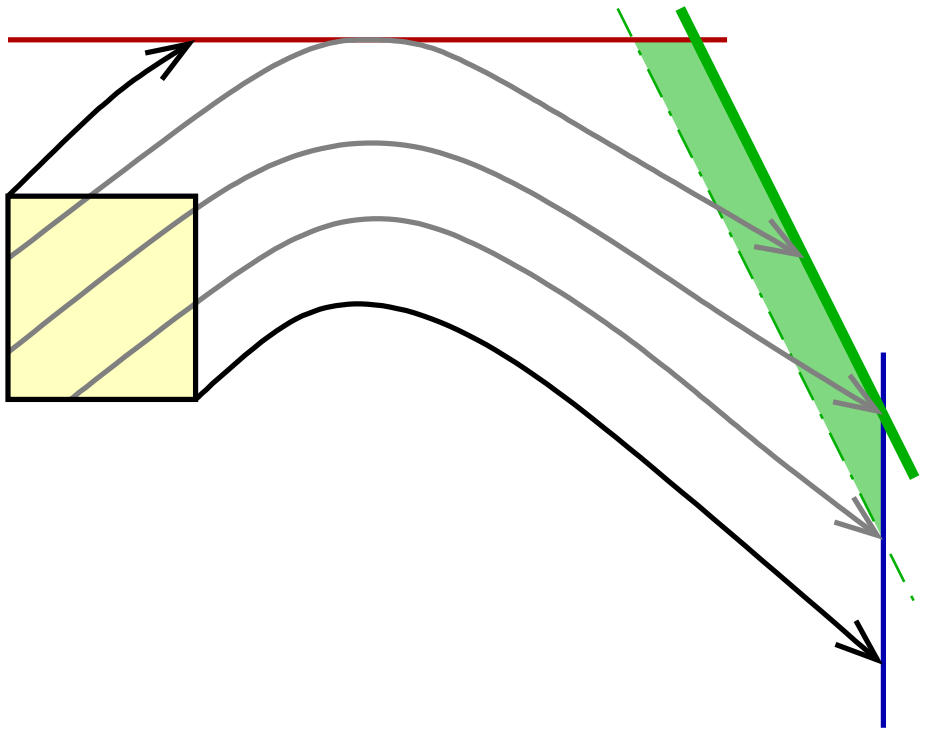

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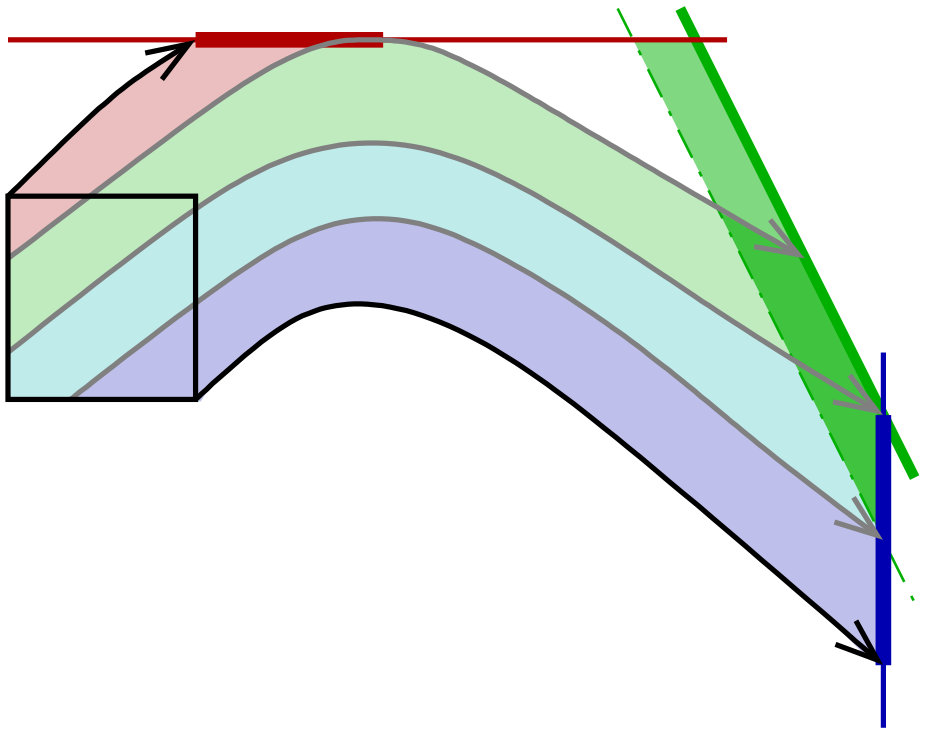

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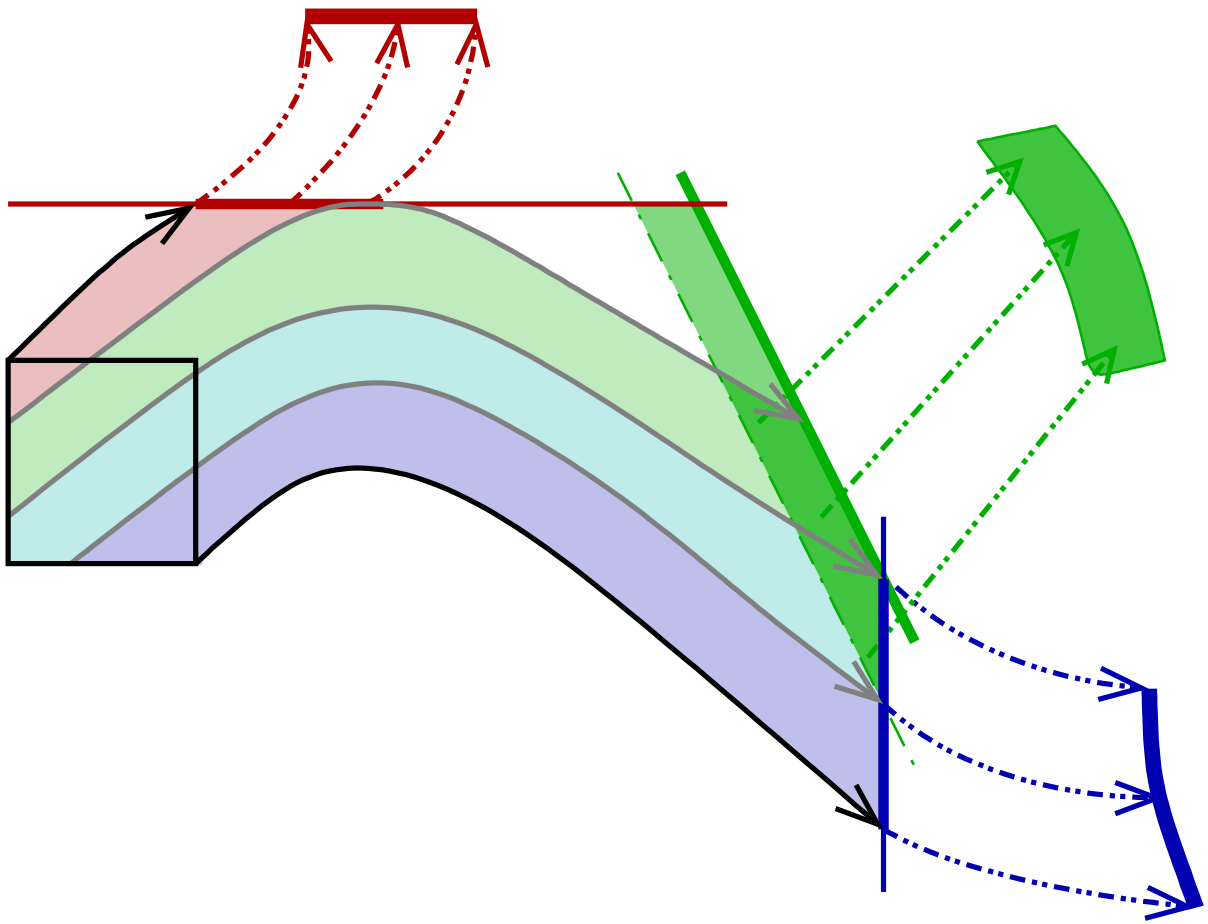

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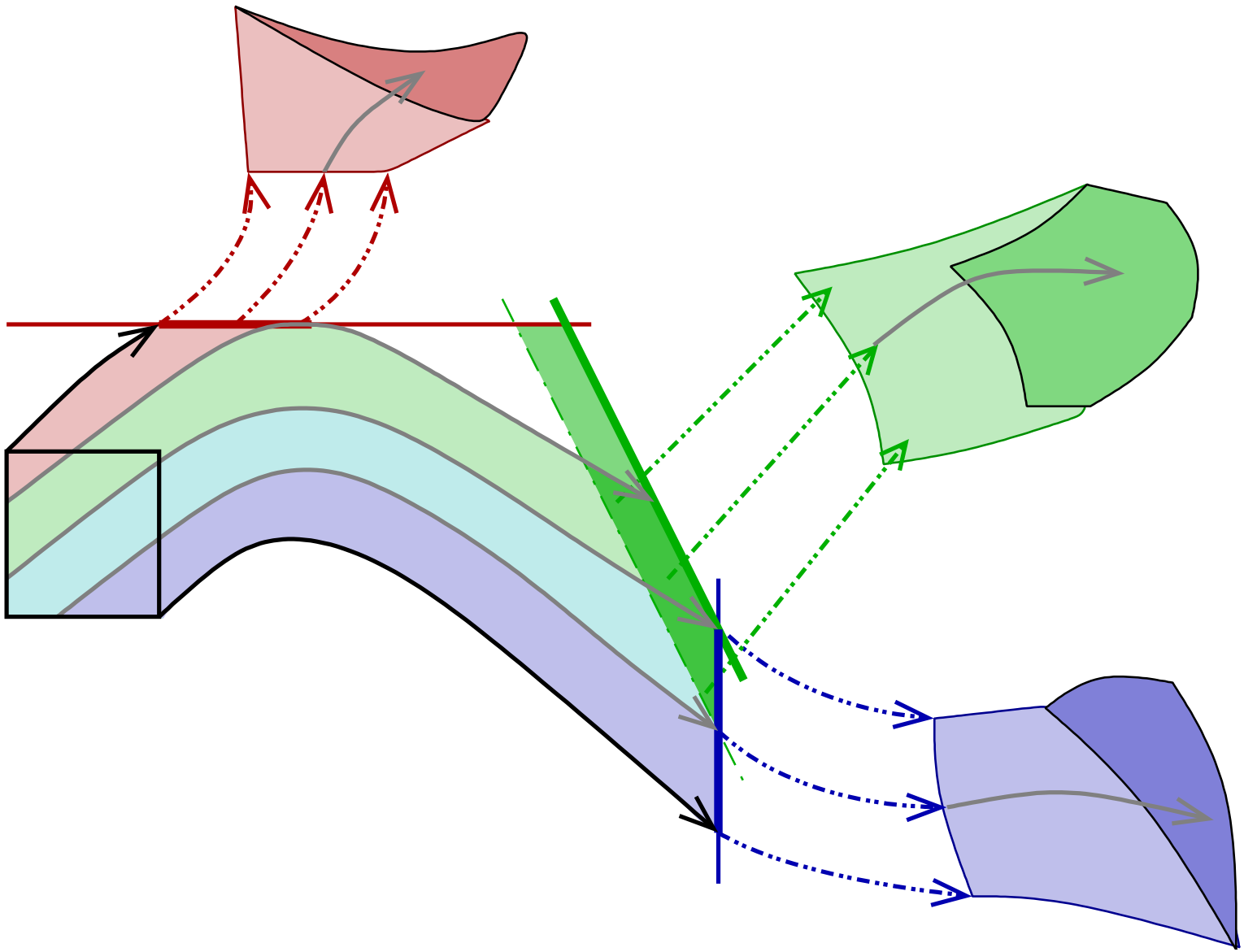

Infinite vs finite time evolution

Infinite-time evolution in practice

A sequence of finite-time evolutions, which terminates if no additional state space can be reached after a while.

Finite time is simple, but may not be usable

Using finite time evolution to verify a system which evolves for infinite time requires the manual identification of a time interval that still gives formal guarantees.

- Example: if the behavior is guaranteed to be periodic, analyze only one period.

In general, to verify some properties of the system we need to evolve the system for infinite time.

Convergence for infinite-time evolution

To obtain convergence, we have two requirements:

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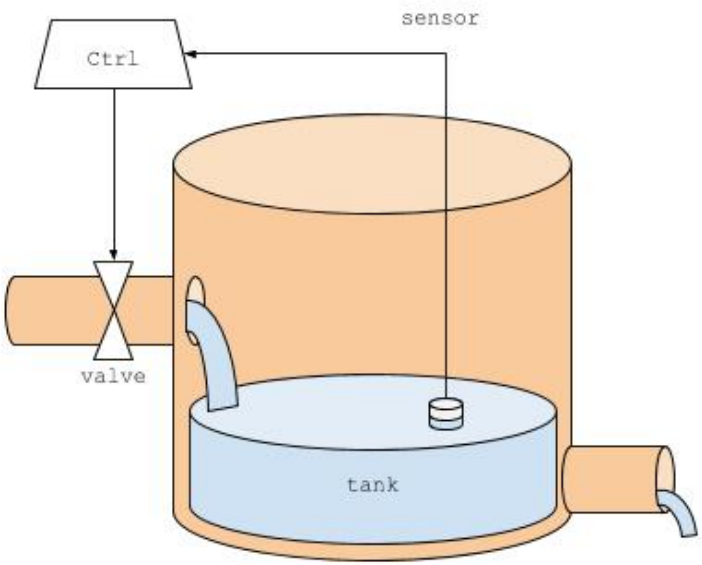
We use Taylor Sets to respect 2., switching temporarily to Grid Sets for 1.

Infinite-time reachability at a glance

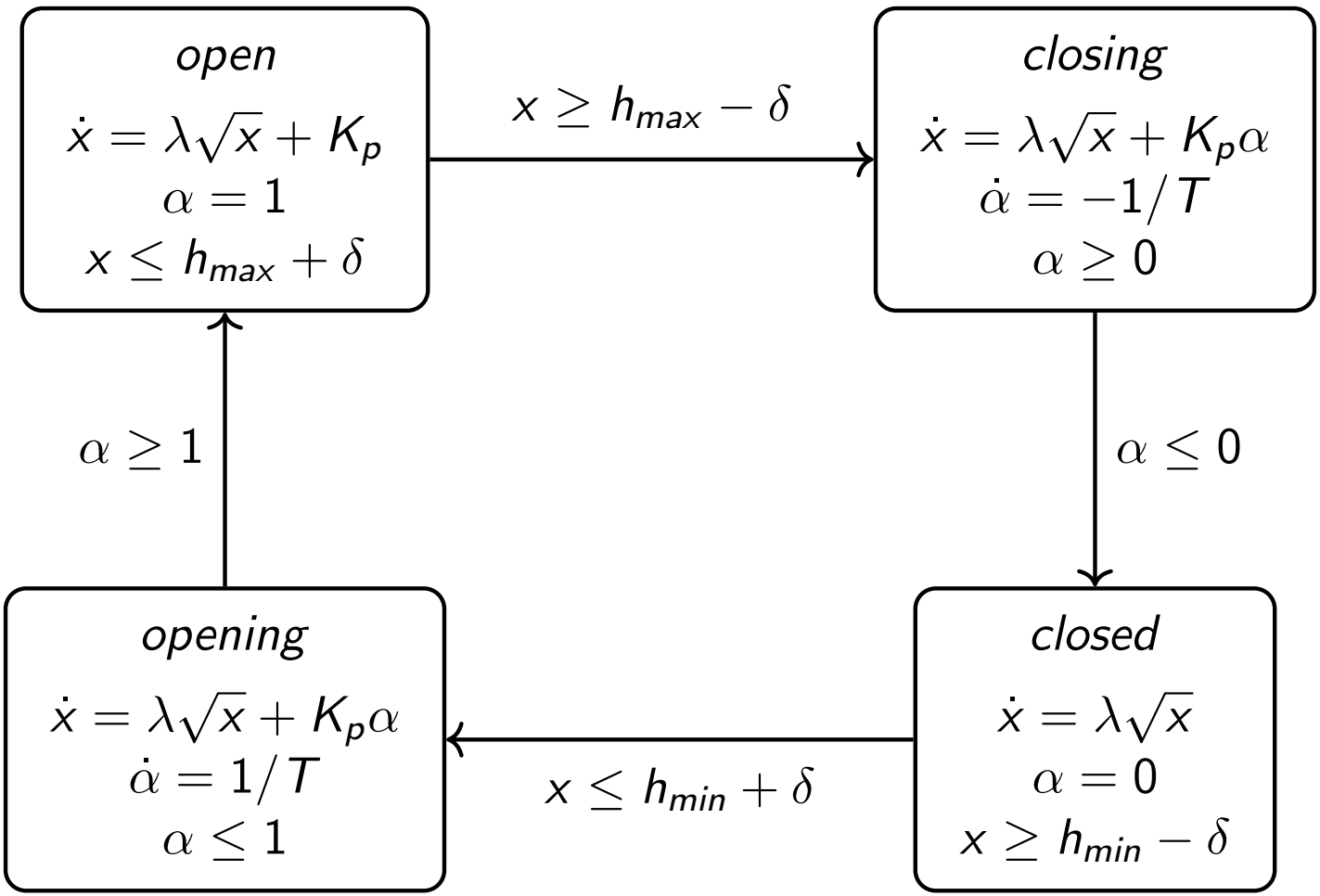
1. Identify a bounding set B to constrain evolution;
2. Approximate non-rigorously the evolution (by points) to identify reasonable lock-to-grid times when the grid set representation should be updated;
3. Compute the finite-time hybrid evolution of the automaton up to the next lock-to-grid time;
4. If the reached set is partially outside the bounding set, stop with failure;
5. If new cells have been found in this iteration, resume from (3);
6. Stop with success.

The watertank example



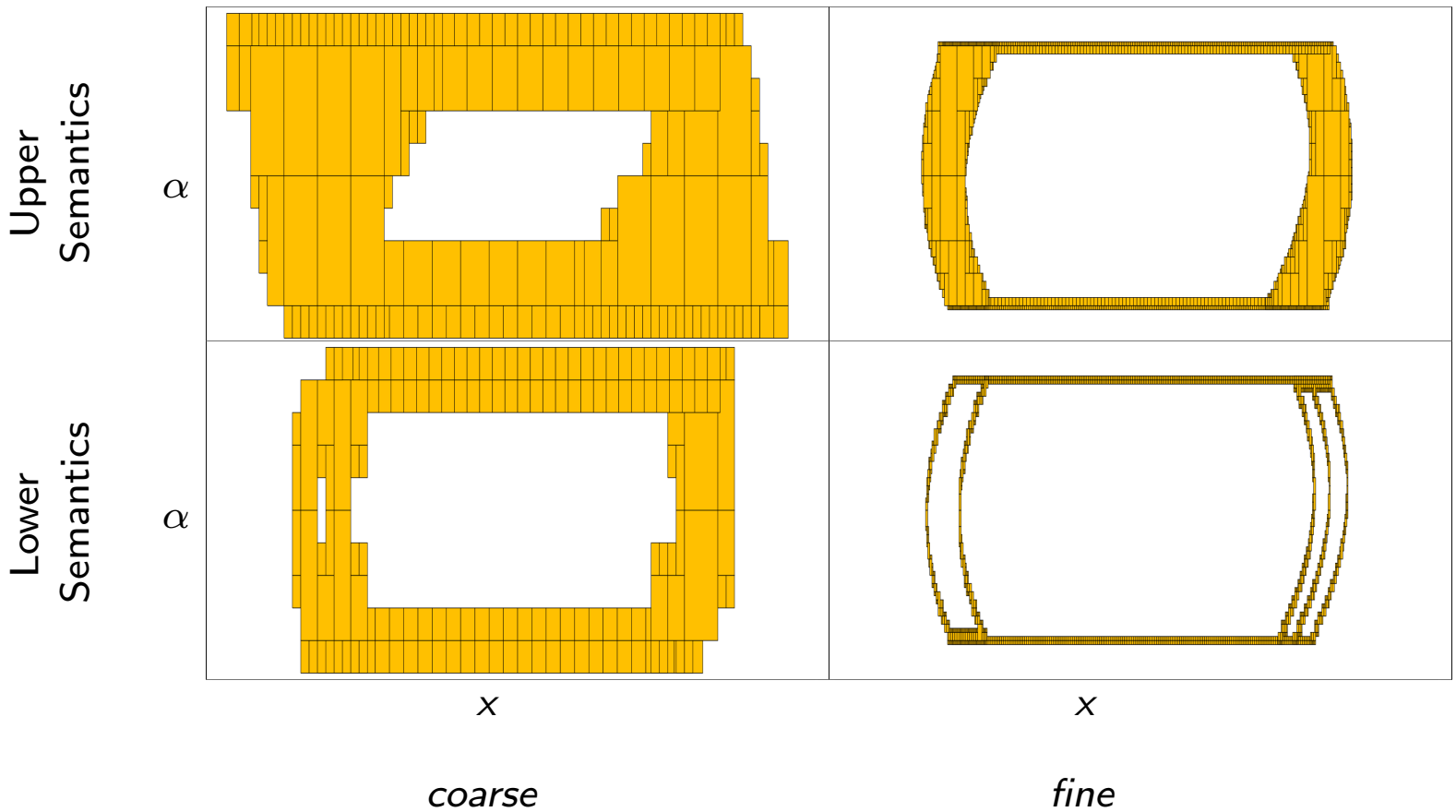
- Outlet flow F_{out} depends on the water level $x(t)$: $F_{out}(t) = \lambda\sqrt{x(t)}$
- Inlet flow F_{in} is controlled by the valve position $\alpha(t)$: $F_{in}(t) = K_p \cdot \alpha(t)$
- The controller senses the water level and sends the appropriate commands to the valve.

The watertank automaton



Reachability results

On the watertank example



Conclusions

Reachability analysis for nonlinear systems is hard

We can compute approximations at arbitrary accuracy, but

- There is no guarantee that properties can be proven with a finite accuracy, and
- There is no representation that is both scalable and accurate

References

- Collins, P.; Bresolin, D.; Geretti, L.; Villa, T. "Computing the evolution of hybrid systems using rigorous function calculus", 4th IFAC Conference on Analysis and Design of Hybrid Systems (ADHS'12), June 2012, pg. 284-290, DOI: 10.3182/20120606-3-NL-3011.00046
- Benvenuti, L; Bresolin, D.; Collins, P.; Ferrari, A.; Geretti, L.; Villa, T. "Assume-guarantee verification of nonlinear hybrid systems with Ariadne", International Journal of Robust and Nonlinear Control, Volume 24, Issue 4, Mar. 2014, pg. 699-724, ISSN: 1049-8923, DOI: 10.1002/RNC.2914